OCR
Oxford Cambridge and RSA

Tuesday 21 June 2022 – Afternoon

A Level Mathematics A

H240/03 Pure Mathematics and Mechanics

Time allowed: 2 hours

You must have:
- the Printed Answer Booklet
- a scientific or graphical calculator

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INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give non-exact numerical answers correct to **3** significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **100**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **12** pages.

ADVICE
- Read each question carefully before you start your answer.

Formulae
A Level Mathematics A (H240)

Arithmetic series

$$S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n\{2a+(n-1)d\}$$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \quad \text{for } |r| < 1$$

Binomial series

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$$

where ${}^nC_r = {}_nC_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$$

Differentiation

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Quotient rule $y = \dfrac{u}{v}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Differentiation from first principles

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

Integration

$$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$$

$$\int \mathrm{f}'(x)\big(\mathrm{f}(x)\big)^n\,\mathrm{d}x = \frac{1}{n+1}\big(\mathrm{f}(x)\big)^{n+1} + c$$

Integration by parts $\displaystyle\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \tfrac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k+\tfrac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0+y_n)+2(y_1+y_2+\ldots+y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0$: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Probability

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$ **or** $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Standard deviation

$\sqrt{\dfrac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\dfrac{\Sigma x^2}{n} - \bar{x}^2}$ **or** $\sqrt{\dfrac{\Sigma f(x-\bar{x})^2}{\Sigma f}} = \sqrt{\dfrac{\Sigma fx^2}{\Sigma f} - \bar{x}^2}$

The binomial distribution

If $X \sim \mathrm{B}(n, p)$ then $\mathrm{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$, mean of X is np, variance of X is $np(1-p)$

Hypothesis test for the mean of a normal distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\bar{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0,1)$

Percentage points of the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that $\mathrm{P}(Z \leqslant z) = p$.

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u+v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = \frac{1}{2}(\mathbf{u}+\mathbf{v})t$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

Section A: Pure Mathematics

Answer **all** the questions.

1 Solve the equation $|2x-3|=9$. **[3]**

2 **(a)** Give full details of the single transformation that transforms the graph of $y=x^3$ to the graph of $y=x^3-8$. **[2]**

The function f is defined by $\mathrm{f}(x)=x^3-8$.

(b) Find an expression for $\mathrm{f}^{-1}(x)$. **[2]**

(c) State how the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$ are related geometrically. **[1]**

3 The points P and Q have coordinates $(2, -5)$ and $(3, 1)$ respectively.

Determine the equation of the circle that has PQ as a diameter. Give your answer in the form $x^2+y^2+ax+by+c=0$, where a, b and c are integers. **[4]**

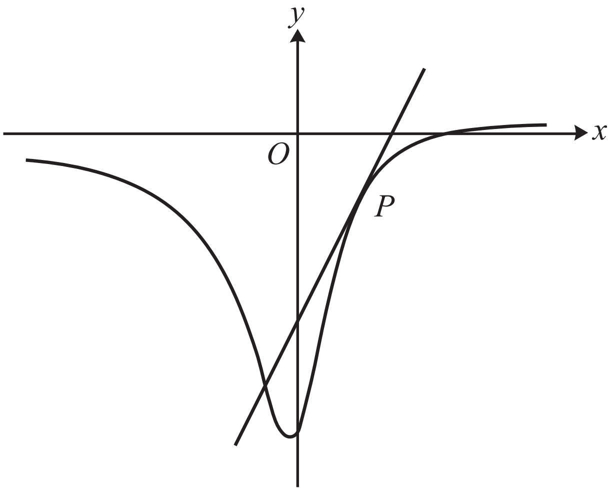
4 The positive integers x, y and z are the first, second and third terms, respectively, of an arithmetic progression with common difference -4.

Also, x, $\frac{15}{y}$ and z are the first, second and third terms, respectively, of a geometric progression.

(a) Show that y satisfies the equation $y^4-16y^2-225=0$. **[4]**

(b) Hence determine the sum to infinity of the geometric progression. **[4]**

5 **In this question you must show detailed reasoning.**



The diagram shows the curve with equation $y = \dfrac{2x-3}{4x^2+1}$. The tangent to the curve at the point P has gradient 2.

(a) Show that the x-coordinate of P satisfies the equation

$$4x^3 + 3x - 3 = 0.$$ **[5]**

(b) Show by calculation that the x-coordinate of P lies between 0.5 and 1. **[2]**

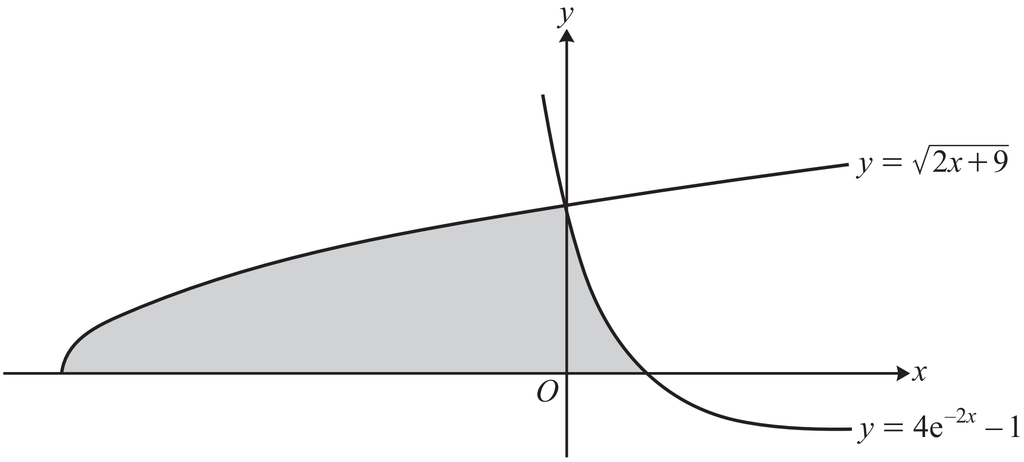
(c) Show that the iteration

$$x_{n+1} = \frac{3-4x_n^3}{3}$$

cannot converge to the x-coordinate of P whatever starting value is used. **[2]**

(d) Use the Newton-Raphson method, with initial value 0.5, to determine the coordinates of P correct to **5** decimal places. **[5]**

6 **In this question you must show detailed reasoning.**



The diagram shows the curves $y = \sqrt{2x+9}$ and $y = 4e^{-2x} - 1$ which intersect on the y-axis. The shaded region is bounded by the curves and the x-axis.

Determine the area of the shaded region, giving your answer in the form $p + q\ln 2$ where p and q are constants to be determined. **[8]**

7 **In this question you must show detailed reasoning.**

(a) Show that the equation $m\sec\theta + 3\cos\theta = 4\sin\theta$ can be expressed in the form

$m\tan^2\theta - 4\tan\theta + (m+3) = 0.$ **[3]**

(b) It is given that there is only one value of θ, for $0 < \theta < \pi$, satisfying the equation $m\sec\theta + 3\cos\theta = 4\sin\theta$.

Given also that m is a negative integer, find this value of θ, correct to **3** significant figures. **[5]**

Section B: Mechanics

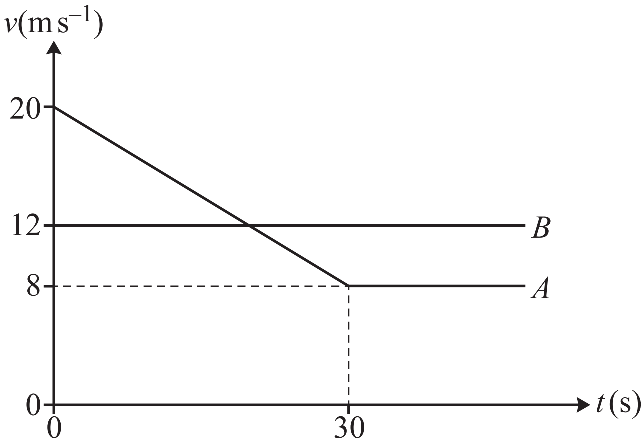
Answer **all** the questions.

8

A child attempts to drag a sledge along horizontal ground by means of a rope attached to the sledge. The rope makes an angle of 15° with the horizontal (see diagram).

Given that the sledge remains at rest and that the frictional force acting on the sledge is 60 N, find the tension in the rope. **[2]**

9



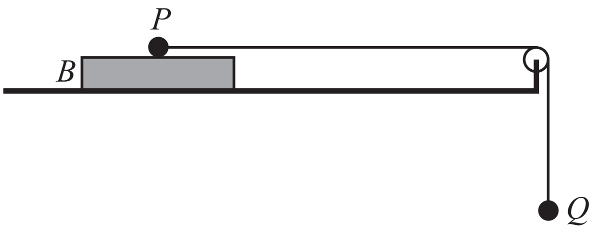
The diagram shows a velocity-time graph representing the motion of two cars A and B which are both travelling along a horizontal straight road. At time $t = 0$, car B, which is travelling with constant speed $12\,\mathrm{m\,s^{-1}}$, is overtaken by car A which has initial speed $20\,\mathrm{m\,s^{-1}}$.

From $t = 0$ car A travels with constant deceleration for 30 seconds. When $t = 30$ the speed of car A is $8\,\mathrm{m\,s^{-1}}$ and the car maintains this speed in its subsequent motion.

(a) Calculate the deceleration of car A. **[2]**

(b) Determine the value of t when B overtakes A. **[4]**

10



A rectangular block B is at rest on a horizontal surface. A particle P of mass 2.5 kg is placed on the upper surface of B. The particle P is attached to one end of a light inextensible string which passes over a smooth fixed pulley. A particle Q of mass 3 kg is attached to the other end of the string and hangs freely below the pulley. The part of the string between P and the pulley is horizontal (see diagram).

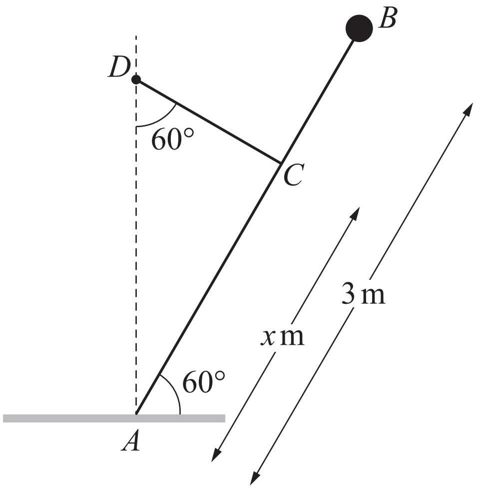
The particles are released from rest with the string taut. It is given that B remains in equilibrium while P moves on the upper surface of B. The tension in the string while P moves on B is 16.8 N.

(a) Find the acceleration of Q while P and B are in contact. **[2]**

(b) Determine the coefficient of friction between P and B. **[3]**

(c) Given that the coefficient of friction between B and the horizontal surface is $\frac{5}{49}$, determine the least possible value for the mass of B. **[3]**

11



A uniform rod AB of mass 4 kg and length 3 m rests in a vertical plane with A on rough horizontal ground.

A particle of mass 1 kg is attached to the rod at B. The rod makes an angle of 60° with the horizontal and is held in limiting equilibrium by a light inextensible string CD. D is a fixed point vertically above A and CD makes an angle of 60° with the vertical. The distance AC is x m (see diagram).

(a) Find, in terms of g and x, the tension in the string. **[3]**

The coefficient of friction between the rod and the ground is $\frac{9\sqrt{3}}{35}$.

(b) Determine the value of x. **[4]**

12 In this question the unit vectors $\mathbf{i}$ and $\mathbf{j}$ are in the directions east and north respectively.

A particle P is moving on a smooth horizontal surface under the action of a single force $\mathbf{F}$ N. At time t seconds, where $t \geqslant 0$, the velocity $\mathbf{v}\,\mathrm{m\,s^{-1}}$ of P, relative to a fixed origin O, is given by

$\mathbf{v} = (1-2t)\mathbf{i} + (2t^2 + t - 13)\mathbf{j}$.

(a) Show that P is never stationary. **[2]**

(b) Find, in terms of $\mathbf{i}$ and $\mathbf{j}$, the acceleration of P at time t. **[1]**

The mass of P is 0.5 kg.

(c) Determine the magnitude of $\mathbf{F}$ when P is moving in the direction of the vector $-2\mathbf{i} + \mathbf{j}$. Give your answer correct to **3** significant figures. **[5]**

When $t = 1$, P is at the point with position vector $\frac{1}{6}\mathbf{j}$.

(d) Determine the bearing of P from O at time $t = 1.5$. **[5]**

13 A small ball B moves in the plane of a fixed horizontal axis Ox, which lies on horizontal ground, and a fixed vertically upwards axis Oy. B is projected from O with a velocity whose components along Ox and Oy are $U\,\mathrm{m\,s^{-1}}$ and $V\,\mathrm{m\,s^{-1}}$, respectively. The units of x and y are metres.

B is modelled as a particle moving freely under gravity.

(a) Show that the path of B has equation $2U^2y = 2UVx - gx^2$. **[3]**

During its motion, B just clears a vertical wall of height $\frac{1}{2}a$ m at a horizontal distance a m from O. B strikes the ground at a horizontal distance $3a$ m beyond the wall.

(b) Determine the angle of projection of B. Give your answer in degrees correct to **3** significant figures. **[5]**

(c) Given that the speed of projection of B is $54.6\,\mathrm{m\,s^{-1}}$, determine the value of a. **[2]**

(d) Hence find the maximum height of B above the ground during its motion. **[3]**

(e) State **one** refinement of the model, other than including air resistance, that would make it more realistic. **[1]**

END OF QUESTION PAPER

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