Oxford Cambridge and RSA

Wednesday 6 October 2021 – Afternoon

A Level Mathematics A

H240/01 Pure Mathematics

Time allowed: 2 hours

You must have:
- the Printed Answer Booklet
- a scientific or graphical calculator

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INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give non-exact numerical answers correct to **3** significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **100**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

Formulae
A Level Mathematics A (H240)

Arithmetic series

$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a+(n-1)d\}$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \quad \text{for } |r| < 1$$

Binomial series

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N})$$

where ${}^nC_r = {}_nC_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1, n \in \mathbb{R})$$

Differentiation

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Quotient rule $y = \dfrac{u}{v}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Differentiation from first principles

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

Integration

$$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$$

$$\int \mathrm{f}'(x)\big(\mathrm{f}(x)\big)^n\,\mathrm{d}x = \frac{1}{n+1}\big(\mathrm{f}(x)\big)^{n+1} + c$$

Integration by parts $\displaystyle\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k + \tfrac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0$: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Probability

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$ **or** $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Standard deviation

$\sqrt{\dfrac{\sum(x-\bar{x})^2}{n}} = \sqrt{\dfrac{\sum x^2}{n} - \bar{x}^2}$ **or** $\sqrt{\dfrac{\sum f(x-\bar{x})^2}{\sum f}} = \sqrt{\dfrac{\sum fx^2}{\sum f} - \bar{x}^2}$

The binomial distribution

If $X \sim \mathrm{B}(n, p)$ then $\mathrm{P}(X = x) = \dbinom{n}{x} p^x (1-p)^{n-x}$, mean of X is np, variance of X is $np(1-p)$

Hypothesis test for the mean of a normal distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\overline{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0, 1)$

Percentage points of the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that $\mathrm{P}(Z \leqslant z) = p$.

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u + v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

Answer **all** the questions.

1 Determine the set of values of k such that the equation $x^2 + 4x + (k+3) = 0$ has **two** distinct real roots. **[4]**

2 Alex is comparing the cost of mobile phone contracts. Contract ***A*** has a set-up cost of £40 and then costs 4p per minute. Contract ***B*** has no set-up cost, does not charge for the first 100 minutes and then costs 6p per minute.

(a) Find an expression for the cost of each of the contracts in terms of m, where m is the number of minutes for which the phone is used and $m > 100$. **[2]**

(b) Hence find the value of m for which both contracts would cost the same. **[2]**

3 It is given that x is proportional to the product of the square of y and the positive square root of z. When $y = 2$ and $z = 9$, $x = 30$.

(a) Write an equation for x in terms of y and z. **[2]**

(b) Find the value of x when $y = 3$ and $z = 25$. **[2]**

4 **In this question you must show detailed reasoning.**

The cubic polynomial $\mathrm{f}(x)$ is defined by $\mathrm{f}(x) = 2x^3 - 3x^2 - 11x + 6$.

(a) Use the factor theorem to show that $(2x-1)$ is a factor of $\mathrm{f}(x)$. **[1]**

(b) Express $\mathrm{f}(x)$ in fully factorised form. **[3]**

(c) Hence solve the equation $2 \times 8^y - 3 \times 4^y - 11 \times 2^y + 6 = 0$. **[3]**

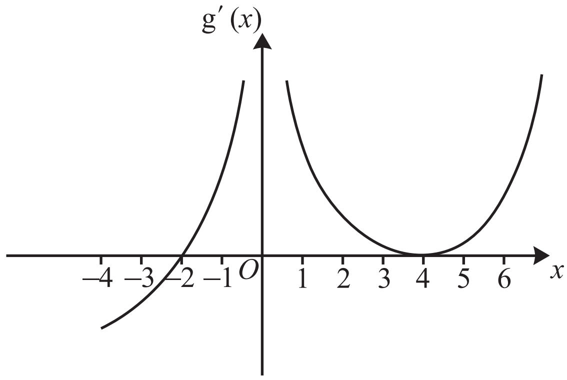
5 **(a)** The graph of the function $y = \mathrm{f}(x)$ passes through the point P with coordinates (2, 6), and is a one-one function. State the coordinates of the point corresponding to P on each of the following curves.

(i) $y = \mathrm{f}(x) + 3$ **[1]**

(ii) $y = 2\mathrm{f}(3x - 1)$ **[2]**

(iii) $y = \mathrm{f}^{-1}(x)$ **[1]**

(b)



The diagram shows part of the graph of $y = \mathrm{g}'(x)$. This is the graph of the gradient function of $y = \mathrm{g}(x)$. The graph intersects the x-axis at $x = -2$ and $x = 4$.

(i) State the x-coordinate of any stationary points on the graph of $y = \mathrm{g}(x)$. **[1]**

(ii) State the set of values of x for which $y = \mathrm{g}(x)$ is a decreasing function. **[1]**

(iii) State the x-coordinate of any points of inflection on the graph of $y = \mathrm{g}(x)$. **[1]**

6 **(a)** Find the first three terms in the expansion of $(8 - 3x)^{\frac{1}{3}}$ in ascending powers of x. **[4]**

(b) State the range of values of x for which the expansion in part **(a)** is valid. **[1]**

(c) Find the coefficient of x^2 in the expansion of $\dfrac{(8 - 3x)^{\frac{1}{3}}}{(1 + 2x)^2}$. **[4]**

7 The curve $y = (x^2 - 2)\ln x$ has one stationary point which is close to $x = 1$.

(a) Show that the x-coordinate of this stationary point satisfies the equation $2x^2 \ln x + x^2 - 2 = 0$. **[2]**

(b) Show that the Newton-Raphson iterative formula for finding the root of the equation in part **(a)** can be written in the form $x_{n+1} = \dfrac{2x_n^2 \ln x_n + 3x_n^2 + 2}{4x_n(\ln x_n + 1)}$. **[4]**

(c) Apply the Newton-Raphson formula with initial value $x_1 = 1$ to find x_2 and x_3. **[1]**

(d) Find the coordinates of this stationary point, giving each coordinate correct to **3** decimal places. **[2]**

8 Functions f and g are defined for $0 \leqslant x \leqslant 2\pi$ by $\mathrm{f}(x) = 2\tan x$ and $\mathrm{g}(x) = \sec x$.

(a) **(i)** State the range of f. **[1]**

(ii) State the range of g. **[1]**

(b) **(i)** Show that $\mathrm{fg}(0.6) = 5.33$, correct to **3** significant figures. **[2]**

(ii) Explain why $\mathrm{f}^{-1}\mathrm{g}(0.6)$ is not defined. **[1]**

(c) **In this question you must show detailed reasoning.**

Solve the equation $(\mathrm{f}(x))^2 + 6\mathrm{g}(x) = 0$. **[5]**

9 A particle moves in the x-y plane so that at time t seconds, where $t \geqslant 0$, its coordinates are given by

$x = \mathrm{e}^{2t} - 4\mathrm{e}^{t} + 3$, $y = 2\mathrm{e}^{-3t}$.

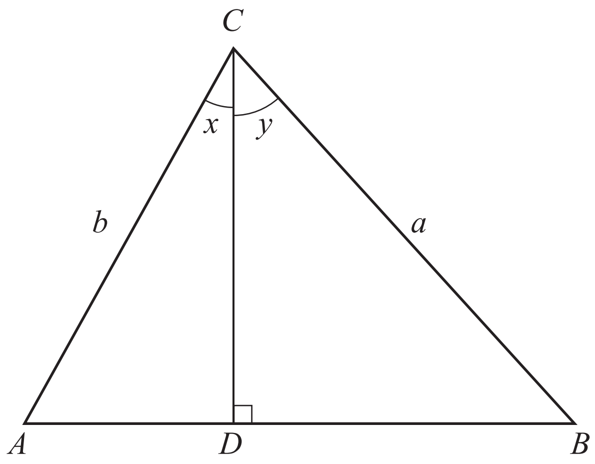
(a) Explain why the path of the particle never crosses the x-axis. **[1]**

(b) Determine the exact values of t when the path of the particle intersects the y-axis. **[2]**

(c) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3}{2\mathrm{e}^{4t} - \mathrm{e}^{5t}}$. **[4]**

(d) Hence find the coordinates of the particle when its path is parallel to the y-axis. **[3]**

10 **(a)**



The diagram shows triangle ABC. The perpendicular from C to AB meets AB at D.

Angle $ACD = x$, angle $DCB = y$, length $BC = a$ and length $AC = b$.

(i) Explain why the length of CD can be written as $a\cos y$. **[1]**

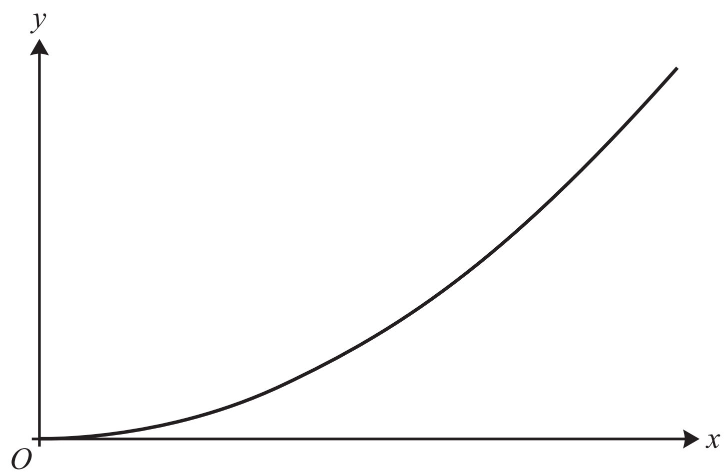
(ii) Show that the area of the triangle ADC is given by $\frac{1}{2}ab\sin x\cos y$. **[1]**

(iii) Hence, or otherwise, show that $\sin(x+y) = \sin x\cos y + \cos x\sin y$. **[4]**

(b) Given that $\sin(30° + \alpha) = \cos(45° - \alpha)$, show that $\tan\alpha = 2 + \sqrt{6} - \sqrt{3} - \sqrt{2}$. **[5]**

11 **(a)** Use the substitution $u^2 = x^2 + 3$ to show that $\int \frac{4x^3}{\sqrt{x^2+3}}\mathrm{d}x = \frac{4}{3}(x^2-6)\sqrt{x^2+3} + c$. **[5]**

(b) **In this question you must show detailed reasoning.**



The graph shows part of the curve $y = \frac{4x^3}{\sqrt{x^2+2}}$.

Find the exact area enclosed by the curve $y = \frac{4x^3}{\sqrt{x^2+3}}$, the normal to this curve at the point $(1, 2)$ and the x-axis. **[7]**

12 A cake is cooling so that, t minutes after it is removed from an oven, its temperature is θ °C.
When the cake is removed from the oven, its temperature is 160 °C. After 10 minutes its temperature has fallen to 125 °C.

(a) In a simple model, the rate of decrease of the temperature of the cake is assumed to be constant.

(i) Write down a differential equation for this model. **[1]**

(ii) Solve this differential equation to find θ in terms of t. **[2]**

(iii) State **one** limitation of this model. **[1]**

(b) In a revised model, the rate of decrease of the temperature of the cake is proportional to the difference between the temperature of the cake and the temperature of the room. The temperature of the room is a constant 20 °C.

(i) Write down a differential equation for this revised model. **[1]**

(ii) Solve this differential equation to find θ in terms of t. **[6]**

(c) The cake can be decorated when its temperature is 25 °C. Find the difference in time between when the two models would predict that the cake can be decorated, giving your answer correct to the nearest minute. **[2]**

END OF QUESTION PAPER